Nonequilibrium transitions for a stochastic globally coupled model

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We report on two globally coupled models driven by noises, and study their nonequilibrium transitions. It is shown that these models both have the symmetry-breaking transition reported by Broeck and co-workers [Phys. Rev. Lett. **73**, 3395 (1994); Phys. Rev. E **49**, 541 (1994)], and a non-symmetry-breaking transition under some circumstances. The former is a second-order phase transition, and the latter is not a phase transition. [S1063-651X(98)02109-6]

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I. INTRODUCTION

Lately there has been an increasing interest in the influence of noises on nonequilibrium transitions and bifurcations for a system with finite or infinite oscillators [1–12]. Usually, two types of coupling are considered. One is local coupling, when each oscillator is influenced only by its neighbors [1–8]. The other is global coupling, when the interaction does not depend on the distance between oscillators [9–15].

In this paper we shall consider two models with infinite globally coupled oscillators driven by noises, and investigate the nonequilibrium transitions for them. In order to explain clearly the transitions for our models, let us first consider the transitions in Refs. [1–12]. In Refs. [1–5,7–12], the transitions are second order, while in Ref. [6] the transition is first order. However, these transitions are accompanied with a breaking of symmetry, and are between the state with the mean field $s=0$ and the state with the mean field $s\neq 0$. For convenience, we call the phase transitions in Refs. [1–10,12] symmetry-breaking mean-field (SBMF) transitions.

II. ADDITIVE NOISE MODEL

We consider a model whose Langevin equations of oscillators are (in dimensionless form)

$$\dot{x}_i=-x_i^3+\epsilon s x_i^2+b x_i+\xi_i(t) \quad (i=1,2,3,\ldots), \tag{1}$$

where $\{\xi_i(t)\}$ are Gaussian white noises with zero mean and correlation function $\langle \xi_i(t)\xi_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$, ϵ is a positive coupling constant, and the mean field $s=\lim_{N\to\infty}(1/N)\Sigma_i^N x_i$, which has appeared in previous work [16,17]. Because of $N\to\infty$, all the oscillators have an identical evolution given by the nonlinear stochastic equation

$$\dot{x}=-x^3+\epsilon s x^2+bx+\xi(t), \tag{2}$$

where $s(t)=\langle x(t)\rangle$, which represents the time-dependent order parameter. The corresponding nonlinear Fokker-Planck equation (NLFPE) is [18,19]

$$\partial_t P(x,t)=\partial_x U_0'(x,s)P(x,t)+D\partial_x^2 P(x,t), \tag{3}$$

in which the prime indicates a derivative with respect to x of the potential $U_0(x,s)=x^4/4-\epsilon s x^3/3-bx^2/2$, and s satisfies $s=\int_{-\infty}^{\infty} xP(x,t)dx$. The functional form of the stationary solution of the NLFPE is obtained as

$$P_{\rm st}(x,s_{\rm st})=Z^{-1}\exp\left[-\frac{U_0(x,s_{\rm st})}{D}\right], \tag{4}$$

where Z is the normalization constant. The potential that shows up in Eq. (4) depends on $s_{\rm st}$, which has to comply with the condition

$$s_{\rm st}=\int_{-\infty}^{\infty} xP_{\rm st}(x,s_{\rm st})dx=F(s_{\rm st}). \tag{5}$$

The solution of this implicit equation yields the dependence of $s_{\rm st}$ with the system parameters. Obviously the trivial solution $s_{\rm st}=0$ always exists. With the appearance of multiple solutions, we can find $s_{\rm st}\neq 0$. Below, we shall investigate the transitions for model (1) according to Eqs. (4) and (5).

A. Case of $b>0$

It can easily be verified that the function $F(s_{\rm st}) =\int_{-\infty}^{\infty} xP_{\rm st}(x,s_{\rm st})dx$ is a smooth, monotonic, and odd function. When $(\partial/\partial s_{\rm st})F(s_{\rm st})\leqslant 1$, the function $F=F(s_{\rm st})$ crosses the function $F=s_{\rm st}$ at $s_{\rm st}=0$ (stable); when $(\partial/\partial s_{\rm st})F(s_{\rm st}) >1$, the function $F=F(s_{\rm st})$ crosses the function $F=s_{\rm st}$ at $s_{\rm st}=0$ (unstable) and $s_{\rm st}=\pm s_{\rm st}^0$ (stable, $s_{\rm st}^0>0$). Thus the condition that the system transits from the state $s_{\rm st}=0$ to the state $s_{\rm st}\neq 0$, or vice versa, is $(\partial/\partial s_{\rm st})F(s_{\rm st})|_{s_{\rm st}=0}=1$, i.e., $F'(s_{\rm st}=0)=1$. Now the system has two stationary states, which are respectively the symmetric bistable state ($s_{\rm st}=0$) and the asymmetric bistable state ($s_{\rm st}\neq 0$). It is clear that the transition happening here is the SBMF one.

The phase transition line determined by $F'(s_{\rm st}=0)=1$ is plotted in Fig. 1(a). Here we set $b=1$. The region below the curve corresponds to the asymmetric bistable state, and that above the curve to the symmetric bistable state. At the phase transition line there is a bifurcation of the probability density. The nonzero value of $s_{\rm st}$ is represented in Fig. 1(b) from Eq. (5) (the order parameter of the phase transition is $m =|s_{\rm st}|$). The figure shows that (a) the transition is second order, since the order parameter changes continuously; and

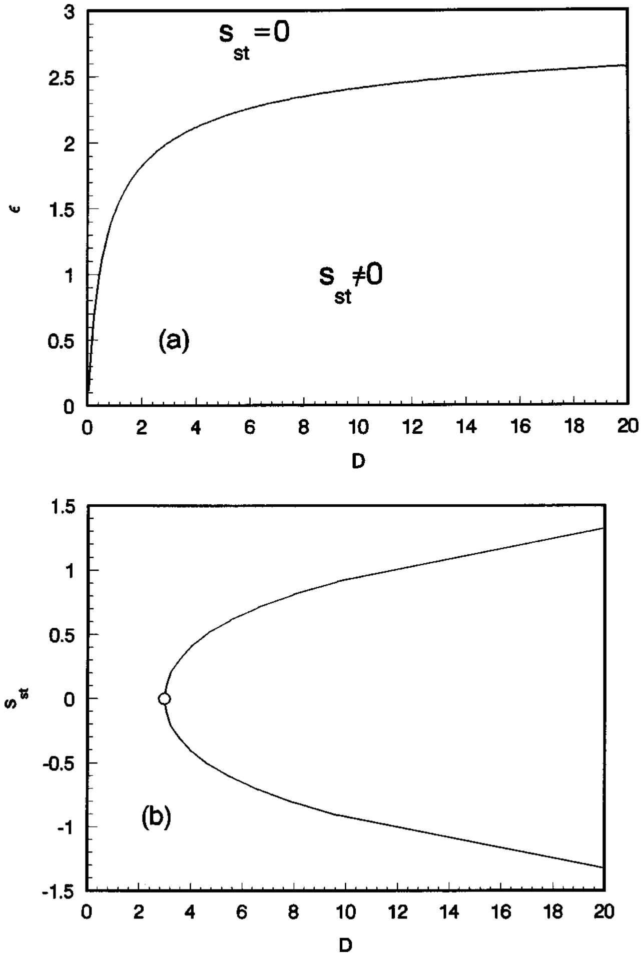

FIG. 1. The SBMF transition line in the ϵ vs D plane, and the stationary mean-field value as a function of D in the case of $b=1$ for model (1). (a) is the phase transition diagram, and (b) is the stationary mean-field diagram.

(2) the part of $s_{\mathrm{st}}>0$ and that of $s_{\mathrm{st}}<0$ are basically symmetric with respect to the D-coordinate axis. The point $s=0$ at the curve which cannot be calculated from Eq. (5) has been marked by a little circle.

B. Case of $b<0$

In this case the system has three stationary states for the potential. The first is the symmetric monostable state ($s_{\mathrm{st}}=0$), the second is the asymmetric monostable state ($s_{\mathrm{st}}\neq 0$), and the third is the asymmetric bistable state ($s_{\mathrm{st}}\neq 0$). When $s_{\mathrm{st}}\neq 0$, there is a critical condition separating the asymmetric monostable state and the asymmetric bistable state. It is $s_{\mathrm{st}}=s_{\mathrm{st}}^{(0)}=\pm(2\sqrt{-b})/\epsilon=\pm 2/\epsilon=\pm s_0$ (see the Appendix; here we set $b=-1$ and $s_0=2/\epsilon$). For convenience, we only consider the case of $s_{\mathrm{st}}>0$ below.

It can be verified that for a given value of ϵ, when the system is in the symmetric monostable state, the function $F=F(s_{\mathrm{st}})$ crosses the function $F=s_{\mathrm{st}}$ only at $s_{\mathrm{st}}=0$ (stable); and when the system is in the asymmetric monostable state the function $F=F(s_{\mathrm{st}})$ crosses the function $F=s_{\mathrm{st}}$ at $s_{\mathrm{st}}=0$ (unstable) and $s_{\mathrm{st}}=m_0$ ($0<m_0<s_0$, stable); and when the system is in the asymmetric bistable state the function $F=F(s_{\mathrm{st}})$ crosses the function $F=s_{\mathrm{st}}$ at $s_{\mathrm{st}}=0$ (unstable), $s_{\mathrm{st}}=s_0$ ($s_0=2/\epsilon$, unstable), and $s_{\mathrm{st}}=m_1$ ($s_0<m_1<\infty$, stable).

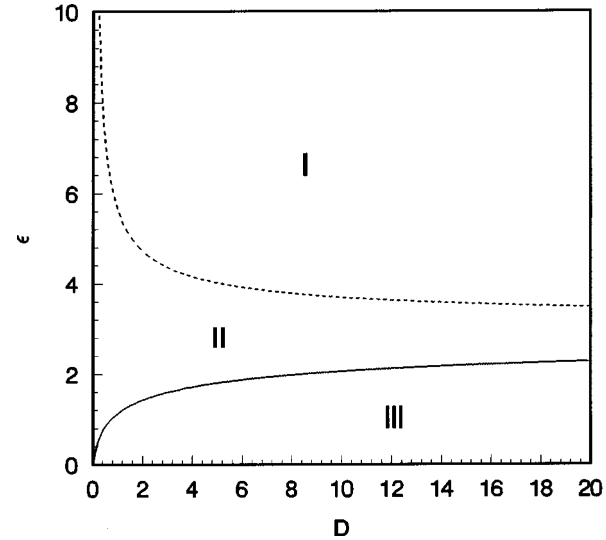

FIG. 2. The SBMF transition line (dashed) and the NSBMF transition line (solid) in the ϵ vs D plane in the case of $b=-1$ for model (1).

It is clear that there are nonequilibrium transitions between the symmetric monostable state and the asymmetric monostable state, and between the asymmetric monostable state and the asymmetric bistable state. The former is the SBMF transition (second order). The latter is a transition without breaking of symmetry. For convenience, we call the latter transition the non-symmetry-breaking mean-field transition (NSBMF transition). The NSBMF transition here is not a phase transition, and does not possess features similar to those observed at the equilibrium phase transitions. Using a similar method as in the case of $b>0$, we can easily verify that the conditions for the SBMF and NSBMF transitions to appear are $(\partial/\partial s_{\mathrm{st}})F(s_{\mathrm{st}})|_{s_{\mathrm{st}}=0}=1$ and $(\partial/\partial s_{\mathrm{st}})F(s_{\mathrm{st}})|_{s_{\mathrm{st}}=s_0}=1$, respectively. If there is a transition between the symmetric monostable state and the asymmetric bistable state, it is first order (since the order parameter changes discontinu-

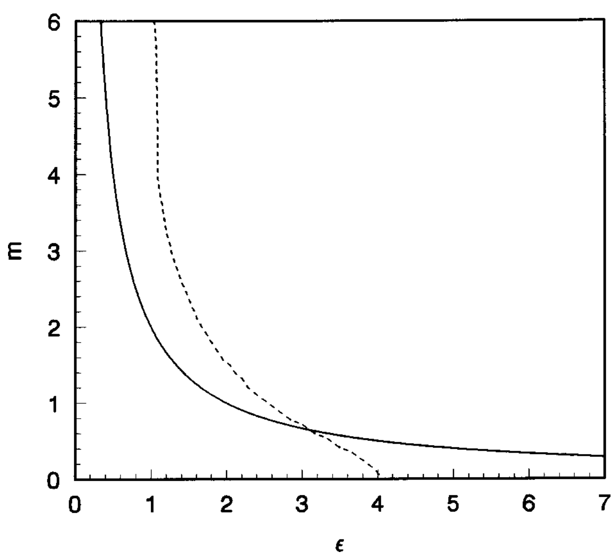

FIG. 3. The positive stationary mean field m (dashed line) as a function of ϵ for model (1). $b=-1$ and $D=5$. The solid line is determined by $m=2/\epsilon$.

ously). However, by the following study we can find that this first-order transition does not exist.

The SBMF and NSBMF transition lines are plotted in Fig. 2. The dashed line is the SBMF and the solid line is the NSBMF. In Fig. 2, region I corresponds to the symmetric monostable state, region II to the asymmetric monostable state, and region III to the asymmetric bistable state. From Fig. 2 we can find that there is not a transition between the symmetric monostable state and the asymmetric bistable state unless the noise strength tends to infinity. From Eq. (5) we plot the positive mean field $m=|s_{\mathrm{st}}|$ (dashed line; it is the order parameter) as a function of the coupling constant ϵ for $D=5$ in Fig. 3. In order to obtain the value of the order parameter when the NSBMF transition occurs, in Fig. 3 we plot the solid line determined by $\epsilon|s_{\mathrm{st}}|=\epsilon m=2$, which is the critical condition separating the asymmetric monostable state and the asymmetric bistable state (see the Appendix). From the figure we can find that the solid line crosses the dashed line at $m\doteq 0.66$. Thus the value of the order parameter is $m\doteq 0.66$ when the NSBMF transition occurs. Now the NSBMF transition is between the state (the asymmetric monostable state) with $0<m\leqslant 0.66$ and the state (the asymmetric bistable state) with $m>0.66$.

III. MULTIPLICATIVE NOISE MODEL

We now consider the case when multiplicative noises exist. Now the stochastic differential equations of oscillators are (in dimensionless form)

$$\dot{x}_i=-x_i^3+\epsilon s x_i^2+b x_i+x_i\eta_i(t)+\xi_i(t) \quad (i=1,2,3,\ldots), \tag{6}$$

where $\{\eta_i(t)\}$ are Gaussian white noises with zero mean, and the correlation functions $\langle\eta_i(t)\eta_j(t')\rangle=2D'\delta_{ij}\delta(t-t')$ and $\langle\eta_i(t)\xi_j(t)\rangle=0$. $\{\xi_i(t)\}$ are the same as those in Eq. (1). The stationary solution of the NLFPE for Eq. (6) is [18,19]

$$P_{\mathrm{st}}(x,s_{\mathrm{st}})=Z'^{-1}\exp\left\{-\left[x^2/2-\epsilon s_{\mathrm{st}}x-\frac{b-D'+D/D'}{2}\ln(x^2+D/D')+\epsilon s_{\mathrm{st}}\sqrt{D/D'}\tan^{-1}(x\sqrt{D'/D})\right]\Big/ D'\right\}, \tag{7}$$

where we drop the subscript i for simplicity since when $N\to\infty$ all the oscillators have an identical evolution, and Z' is the normalization constant. In this case we define an effective potential $U_{\mathrm{FP}}(x,s_{\mathrm{st}})$, which is written as

$$U_{\mathrm{FP}}(x,s_{\mathrm{st}})=-D'\ln P_{\mathrm{st}}(x,s_{\mathrm{st}}). \tag{8}$$

The effective potential $U_{\mathrm{FP}}(x,s_{\mathrm{st}})$ for Eq. (6) is different from the potential $U_0(x,s_{\mathrm{st}})$ in the additive noise case. But the equations for their extrema are basically the same. The equation of extrema for Eq. (8) is

$$x^3-\epsilon s_{\mathrm{st}}x^2-(b-D')x=0 \tag{9}$$

[the equation of extrema for $U_0(x,s_{\mathrm{st}})$ is $x^3-\epsilon s_{\mathrm{st}}x^2-bx=0$].

Obviously the trivial solution $s_{\mathrm{st}}=0$ always exists [for $s_{\mathrm{st}}=0$, $P_{\mathrm{st}}(x,s_{\mathrm{st}})$ is symmetric]. With the appearance of multiple solutions, we can find $s_{\mathrm{st}}\neq 0$ [the symmetry of $P_{\mathrm{st}}(x,s_{\mathrm{st}})$ is broken]. In the following we first consider the case when $b>0$.

From Eq. (8) and its critical condition, one can find that if $D'<b$, when $s_{\mathrm{st}}=0$ the system is in the symmetric bistable state for the potential, while when $s_{\mathrm{st}}\neq 0$ the system is in the asymmetric bistable state; if $D'>b$, when $s_{\mathrm{st}}=0$ the system is in the symmetric monostable state, while when $s_{\mathrm{st}}\neq 0$ the system is in the asymmetric monostable state or the asymmetric bistable state. It is clear that when $D'<b$ the transition is the SBMF one, while when $D'>b$ the transition is the SBMF or NSBMF one. Now the SBMF transition is a second-order phase transition; the NSBMF transition is not a phase transition.

In Fig. 4, we plot the SBMF transition lines (dashed) and the NSBMF transition lines (solid) (here we set $b=3$). The figure shows that the system has four stationary states which are, respectively, the symmetric monostable state (region I), the asymmetric monostable state (the region II), the symmetric bistable state (region III), and the asymmetric bistable state (region IV). In Fig. 4, lines 1, 2, and 3, respectively, are determined by $F'(s_{\mathrm{st}}=0)=1$ [see Sec. II; now Eq. (5) is still applicable], $F'(s_{\mathrm{st}}=|s_{\mathrm{st}}^{(0)}|)=1$ (see Sec. II; now $|s_{\mathrm{st}}^{(0)}|=2\sqrt{D'-b}/\epsilon$ for $b=3$), and $D'=3$. The positive mean field $m=|s_{\mathrm{st}}|$ (the order parameter) is represented in Fig. 5

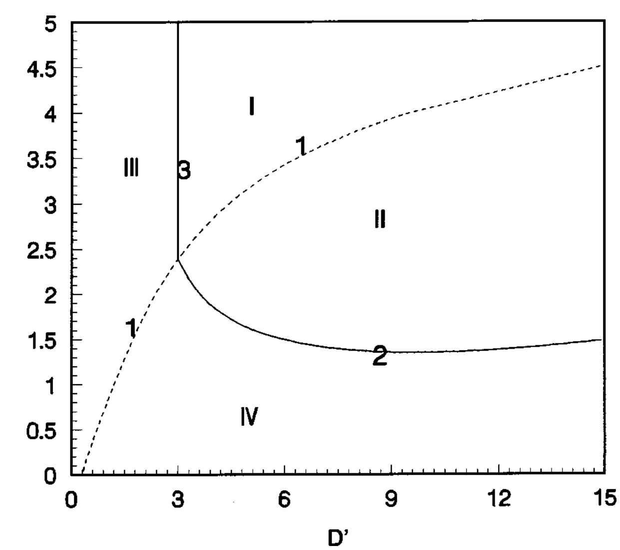

FIG. 4. The SBMF transition line (dashed) and the NSBMF transition line (solid) in the ϵ versus D' plane for the model (6). $b=3$ and $D=1$. Lines 1, 2, and 3, respectively, are determined by $F'(s_{\mathrm{st}}=0)=1$ [now Eq. (5) is still applicable], $F'(s_{\mathrm{st}}=|s_{\mathrm{st}}^{(0)}|)=1$ (now $|s_{\mathrm{st}}^{(0)}|=2\sqrt{D'-b}/\epsilon$ for $b=3$), and $D'=3$.

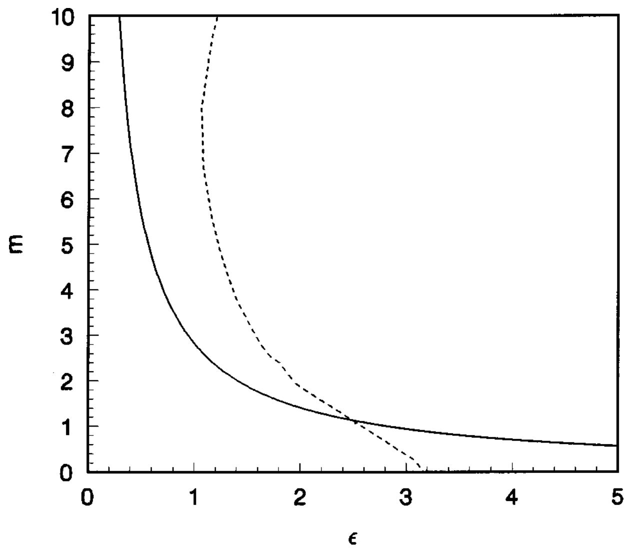

FIG. 5. The positive stationary mean field (dashed line) as a function of ϵ for model (6). $b=3$, $D'=5$, and $D=1$. The solid line is determined by $m=(2\sqrt{D'-b})/\epsilon$ for $D'=5$ and $b=3$.

(dashed line) for a function of ϵ [from Eq. (5)] when $D=1$ and $D'=5$. In order to obtain the value of the order parameter when the NSBMF transition happens, in Fig. 5, we plot the solid line determined by $\epsilon|s_{st}|=\epsilon m=2\sqrt{D'-b}=2\sqrt{5-3}=2\sqrt{2}$, which is the critical condition separating the asymmetric monostable state and the asymmetric bistable state. From this figure, we can obtain the value of the order parameter $m \doteq 1.18$ when the NSBMF transition occurs. Now the NSBMF transition is between the state with $0<m\leqslant 1.18$ and the state with $m>1.18$. If $b<0$, the transition is similar to that in the case of $b>0$ when $D'>b$.

IV. CONCLUSION AND DISCUSSION

In conclusion, we have reported two globally coupled models driven by noises. These models involve both the SBMF and NSBMF transitions under some circumstances. The former transition (SBMF) is second order, and possesses features similar to those observed at the second-order equilibrium phase transitions: divergence of the correlation length and of the susceptibility, critical slowing down, and scaling behavior. The latter transition (NSBMF) is not a phase transition, and does not display features similar to those observed at the equilibrium phase transitions.

The system considered here consists of an infinite number of globally coupled oscillators driven by noises. When the oscillators are finite, the features of the system will change. For example, in Ref. [9], when the oscillators are finite, the system has a transition between the state with zero mean field and the state with nonzero mean field, while when the oscillators are infinite no transition happens in the system. Thus in our paper the case when the oscillators are finite remains to be studied. In addition, what we have studied in the paper is globally coupled. As for the local coupling, a detailed theory is under study.

APPENDIX

In the case of $b<0$ for model (1), if $s_{st}\neq 0$ the system has two stationary states, i.e., the asymmetric monostable state and the asymmetric bistable state. Below, we try to find the critical condition separating the asymmetric monostable state and the asymmetric bistable state.

Let $(\partial/\partial x)U_0(x,s_{st})=0$; we have

$$x^3-\epsilon s_{st}x^2-bx=0.$$

The solutions of this equation are

$$x_1=0,\quad x_2=\frac{\epsilon s+\sqrt{\epsilon^2 s^2+4b}}{2},\quad x_3=\frac{\epsilon s-\sqrt{\epsilon^2 s^2+4b}}{2}.$$

By analysis we can find that when $\epsilon^2 s^2+4b>0$, x_1, x_2, and x_3 are the extreme values for the potential, and the system is in the asymmetric bistable state; however when $\epsilon^2 s^2+4b\leqslant 0$, only $x_1=0$ is the extreme value for the potential, and the system is in the asymmetric monostable state. Thus the critical condition separating the asymmetric monostable state and the asymmetric bistable state is $\epsilon^2 s_{st}^2+4b=0$, i.e., $s_{st}^{(0)}=\pm 2/\epsilon$ (here we set $b=-1$).

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